1) $|\phi_L\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\phi_R\rangle \to \begin{pmatrix}0\\1\end{pmatrix}$

$\langle\phi_L|\hat{H}|\phi_L\rangle = \langle\phi_R|\hat{H}|\phi_R\rangle = \mathcal{E}$ $\quad \langle\phi_L|\hat{H}|\phi_R\rangle = c.c. = T<0$

a)

$$\hat{H} \to \begin{pmatrix}\mathcal{E} & T\\ T & \mathcal{E}\end{pmatrix}$$

$$(\mathcal{E}-\epsilon)^2 - T^2 = 0$$

$$\mathcal{E}-\epsilon = \pm T$$

$$\boxed{\epsilon = \mathcal{E} \mp T}$$

eigen values

$$\begin{pmatrix}\mathcal{E} & T\\ T & \mathcal{E}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = (\mathcal{E}\mp T)\begin{pmatrix}a\\b\end{pmatrix} \Rightarrow \begin{matrix}\mathcal{E}a + Tb = (\mathcal{E}\mp T)a\\ Ta + \mathcal{E}b = (\mathcal{E}\mp T)b\end{matrix}$$

$$a = \mp b$$

Eigenvalue	$\mathcal{E}+T$	$\mathcal{E}-T$
Eigen vector	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$
Eigenstate	$|\phi_g\rangle = \frac{1}{\sqrt{2}}(|\phi_L\rangle + |\phi_R\rangle)$	$\frac{1}{\sqrt{2}}(|\phi_L\rangle - |\phi_R\rangle) = |\phi_e\rangle$
	ground state	excited state

$$\mathcal{E}+T < \mathcal{E}-T \quad \text{for } T<0$$

b) $\hat{P} \equiv |\phi_L\rangle\langle\phi_R| + |\phi_R\rangle\langle\phi_L|$

$$\hat{P}|\phi_g\rangle = \Big(|\phi_L\rangle\langle\phi_R| + |\phi_R\rangle\langle\phi_L|\Big)\frac{1}{\sqrt{2}}\Big(|\phi_L\rangle + |\phi_R\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}|\phi_L\rangle + \frac{1}{\sqrt{2}}|\phi_R\rangle \Rightarrow \boxed{\hat{P}|\phi_g\rangle = |\phi_g\rangle}$$

$$\hat{P}|\phi_e\rangle = -\frac{1}{\sqrt{2}}|\phi_L\rangle + \frac{1}{\sqrt{2}}|\phi_R\rangle \Rightarrow \boxed{\hat{P}|\phi_g\rangle = -|\phi_e\rangle}$$

$|\phi_g\rangle$ and $|\phi_e\rangle$ are eigenstates of $\hat{P}$ (eigenv. ± 1).

$$\hat{P} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$[\hat{H}, \hat{P}] = 0$
since they share the same basis.

c) The state after the measurement is $|\phi_L\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$.

$$\langle \hat{A}(t)\rangle = \langle \phi_L(t)|\hat{A}|\phi_L(t)\rangle$$

$$|\phi_L\rangle = \frac{1}{\sqrt{2}}\left(|\phi_g\rangle + |\phi_e\rangle\right)$$

$$|\phi_L(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\epsilon_g t/\hbar}|\phi_g\rangle + e^{-i\epsilon_e t/\hbar}|\phi_e\rangle\right) \quad \text{with } \begin{array}{l}\epsilon_g = \varepsilon + T\\ \epsilon_e = \varepsilon - T\end{array}$$

$$\hat{A}|\phi_g\rangle = -a|\phi_e\rangle \qquad \hat{A}|\phi_e\rangle = -a|\phi_g\rangle$$

$$\langle \hat{A}(t)\rangle = \frac{1}{\sqrt{2}}\left[\langle\phi_g|e^{i\epsilon_g t/\hbar} + \langle\phi_e|e^{i\epsilon_e t/\hbar}\right]\hat{A}\,\frac{1}{\sqrt{2}}\left[e^{-i\epsilon_g t/\hbar}|\phi_g\rangle + e^{-i\epsilon_e t/\hbar}|\phi_e\rangle\right]$$

$$= -\frac{a}{2}\left[\langle\phi_g|e^{i\epsilon_g t/\hbar} + \langle\phi_e|e^{i\epsilon_e t/\hbar}\right]\left[e^{-i\epsilon_g t/\hbar}|\phi_e\rangle + e^{-i\epsilon_e t/\hbar}|\phi_g\rangle\right]$$

$$= -\frac{a}{2}\left(e^{-i(\epsilon_e - \epsilon_g)t/\hbar} + e^{i(\epsilon_e-\epsilon_g)t/\hbar}\right)$$

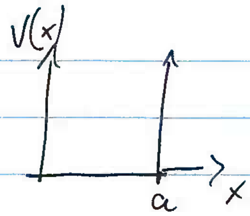
$$\boxed{\langle \hat{A}(t)\rangle = -a\cos\left[\frac{(\epsilon_e-\epsilon_g)}{\hbar}t\right] = -a\cos\left[\frac{-2T}{\hbar}t\right]}$$

d) Ground state: Both electrons at $|\phi_g\rangle$ at spin singlet state

$$|\text{ground}\rangle = \left(|\phi_g\rangle|\phi_g\rangle\right)|\chi_{singlet}\rangle$$

$$|\chi_{singlet}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

(2) $V(x) = \begin{cases} 0 & , \quad 0 < x < a \\ \infty & , \quad \text{otherwise} \end{cases}$



a) $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = E\psi$ (inside well)

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \Rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi \qquad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

Solutions: $\psi(x) = A\sin(kx) + B\cos(kx)$

Boundary cond. $\psi(x=0) = 0 \Rightarrow B = 0$

$\psi(x=a) = 0 \Rightarrow A\sin(ka) = 0$

$ka = n\pi \Rightarrow k = \frac{n\pi}{a}$

$\frac{n\pi}{a} = \frac{\sqrt{2mE}}{\hbar}$

$$\boxed{\psi_n(x) = A\sin\left(\frac{n\pi}{a}x\right) \qquad E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}}$$

b) Ground state : $\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$

$\langle x \rangle = \frac{a}{2}$ since the well is symmetric around $a/2$ with $|\psi_1|^2$ peaking at this value.

$\langle p \rangle = 0$ since particle is trapped in the well.

also $\int_0^a \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{a}\right) dx = 0$

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c) $\langle x \rangle$ and $\langle p \rangle$ do not change in time since we are considering a stationary state.

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d) $a \rightarrow 2a$

$\psi_n' = \sqrt{\frac{2}{(2a)}} \sin\left(\frac{n\pi x}{(2a)}\right)$

$E_n' = \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}$

e) $\Psi(x, t=0) \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) & \text{, for } 0 \le x \le a \\ 0 & \text{, otherwise} \end{cases}$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \Psi_n' e^{-i\varepsilon_n' t/\hbar}$$

with Ψ_n' and ε_n' specified in (d).

$$C_n = \int_{-\infty}^{\infty} \psi_n'^{*} \, \Psi(x, t=0) \, dx$$

$$C_n = \int_0^a \left[\sqrt{\frac{2}{(2a)}} \sin\left(\frac{n\pi x}{2a}\right)\right]\left[\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)\right] dx$$

$$C_n = \int_0^a \frac{2}{a\sqrt{2}} \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) dx$$

3 a) $[L_z, x] = i\hbar y$

$\vec{L} = \vec{r} \times \vec{p} \rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \Rightarrow L_z = (xp_y - yp_x)$

$[L_z, x] = [xp_y, x] - [yp_x, x]$

$= xp_y x - xxp_y - yp_x x + xyp_x$

$= xxp_y - xxp_y - y(p_x x - xp_x)$

$= y(xp_x - p_x x) = y[x, p_x] \Rightarrow \boxed{[L_z, x] = i\hbar y}$

b) $Y_1^0(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$

$L_\pm = L_x \pm i L_y \Rightarrow$ find $Y_1^{\pm 1}$

$Y_1^{\pm 1} = L_\pm Y_1^0$

$$L_\pm = -i\hbar \left\{ -\sin\phi\left(\frac{\partial}{\partial\theta}\right) - \cos\phi\cot\theta\left(\frac{\partial}{\partial\phi}\right) \pm i\cos\phi\left(\frac{\partial}{\partial\theta}\right) \mp i\sin\phi\cot\theta\left(\frac{\partial}{\partial\phi}\right) \right\}$$

$$L_\pm = i\hbar \left\{ \left[\sin\phi \mp i\cos\phi\right]\left(\frac{\partial}{\partial\theta}\right) + \cot\theta\left[\cos\phi \pm i\sin\phi\right]\left(\frac{\partial}{\partial\phi}\right) \right\}$$

$$L_\pm = \hbar \left\{ \left[i\sin\phi \pm \cos\phi\right]\left(\frac{\partial}{\partial\theta}\right) + \cot\theta\left[i\cos\phi \mp \sin\phi\right]\left(\frac{\partial}{\partial\phi}\right) \right\}$$

$$L_\pm Y_1^0 = Y_1^{\pm 1} = -\hbar\left(\frac{3}{4\pi}\right)^{1/2}\left[i\sin\phi \pm \cos\phi\right](\sin\theta)$$

$$= \mp\hbar\left(\frac{3}{4\pi}\right)^{1/2} e^{\pm i\phi}\sin\theta$$

c) Possible values are $\pm \hbar/2$

d) $\langle\Psi|\Psi\rangle = \left[\cos(\theta/2)\langle\uparrow| + e^{-i\phi}\sin(\theta/2)\langle\downarrow|\right]\left[\cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\frac{\theta}{2})|\downarrow\rangle\right]$

$$= \cos^2\left(\frac{\theta}{2}\right)\langle\uparrow|\uparrow\rangle + e^{-i\phi}\sin(\theta/2)\cos(\theta/2)\langle\downarrow|\uparrow\rangle +$$

$$+ \cos(\theta/2)e^{i\phi}\sin(\theta/2)\langle\uparrow|\downarrow\rangle + e^{-i\phi+i\phi}\sin^2(\theta/2)\langle\downarrow|\downarrow\rangle$$

$$= \cos^2\left(\frac{\theta}{2}\right) + 0 + 0 + \sin^2(\theta/2)$$

$$\langle\Psi|\Psi\rangle = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1 \quad //$$

e) Possible outcomes are $\pm\hbar/2$

Probabilities given by:

$+\hbar/2$: $|\langle\uparrow|\Psi\rangle|^2$ & $-\hbar/2$: $|\langle\downarrow|\Psi\rangle|^2$

$|\langle\uparrow|\Psi\rangle|^2 = \cos^2\left(\frac{\theta}{2}\right)$ $\qquad$ $|\langle\downarrow|\Psi\rangle|^2 = \sin^2(\theta/2)$

$= \cos^2(30^\circ) = 3/4$ $\qquad$ $= \sin^2(30^\circ) = 1/4$